

# The Existence of Local Minima in Local-Minimum-Free Potential Surfaces

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## Abstract

Existing approaches to potential field based navigation, such as (Khatib, 1986) and (Rimon and Koditschek, 1992), have traditionally seen the local minimum problem as the only significant obstacle. This is because they have concentrated on the problem of ‘classical’ local minima, characterised by a positive definite Hessian.

This paper demonstrates, via the notion of ‘saddle minimum’, that the navigational problems associated with local minima can also arise in connection with points other than classical local minima. The existing approaches, in concentrating on the classical definition, do not acknowledge or address such local minimum problems.

## 1. Introduction

This paper is aimed at researchers interested in agent navigation, particularly roboticists who use simulation to test their techniques, or researchers seeking to re-apply robotic navigation techniques into virtual world scenarios such as computer games or computer animation in films. This paper may be of particular interest to readers interested in potential field based navigation.

Potential field based navigation (Khatib, 1986) employs the metaphor of a virtual landscape that an agent (typically a robot or computer game character) moves across, when trying to navigate to a goal. Typically, the landscape is shaped so that the gradient of the landscape acts like a force pulling the agent towards its goal, while simultaneously pushing it away from obstacles that lie between it and the goal. The landscape and its gradient are usually modelled using mathematical functions.

Like most other potential field based approaches in AI, this technique is blighted by the Local Minimum Problem (LMP), particularly when using steepest gradient descent to direct travel across the potential field surface.

“...Dealing with local minima is *the* major issue that one has to face in designing a planner based on this approach.” (Latombe, 1991).

Local minima exist when the steepest gradient at all points on the potential field surrounding some point X directs the agent back towards X, and X is not the global minimum position representing the goal of the navigation. These local minima have the effect of causing navigation to fail, as the steepest descent heuristic will not produce any further progress towards the goal when a local minimum is encountered.

Some techniques (Latombe, 1991), try to overcome the LMP by avoiding using the steepest gradient descent heuristic directly. Instead, another technique replaces steepest gradient descent behaviour when local minima are discovered. This strategy involves identifying when the agent’s current location within the potential field is at, or is near a local minimum (either by using mathematical analysis of the surface properties such as the Hessian, or by using heuristics to guess when a minimum might have been reached). When a minimum is identified, backtracking or random motion can be employed instead of steepest gradient descent in order to try and escape the minimum, allowing navigation to continue.

Another family of navigation approaches (Rimon and Koditschek, 1992) (Connolly et al., 1990) (Connolly and Grupen, 1992) (Kim and Khosla, 1992) (Latombe, 1991) seek to overcome the LMP by attempting to construct potential fields that simply do not have local minima in the first place. ‘Harmonic Potential Fields’ are the best known example of this technique. The navigational potential field is defined to be created solely from harmonic functions (those functions containing only saddle points as their stationary points, and not local minima) representing goals and obstacles. The resulting navigational field also lacks local minima. Steepest gradient descent can then be used on such a surface, as there are no longer any local minima on the surface to cause navigation to fail.

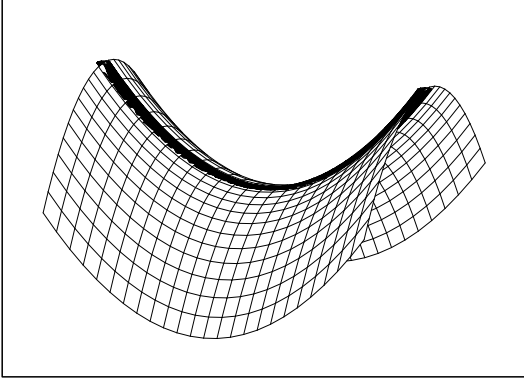


Figure 1: Saddle Point of  $f(x, y) = x^2 - y^2$ . The subspace along which travel can be trapped in a local minimum is marked.

“We use harmonic functions which completely eliminate local minima even for a cluttered environment.” (Kim and Khosla, 1992).

An excellent source of information on all types of approach taken to overcome the LMP can be found in (Latombe, 1991).

This paper will illustrate a type of ‘hidden local minimum’ that may cause problems for both strategies for dealing with the LMP, under certain conditions.

## 2. The Saddle Minimum Problem

The techniques of ‘local minimum detection’ and ‘local-minimum-free potential fields’ have been applied in both real world and simulated robotic navigation experiments. However, a subtle flaw can be exhibited when the navigation heuristics are employed in experimental simulation, or when they are re-applied to the problem domain of virtual-world navigation (i.e. computer games, computer animated films). This flaw is the saddle minimum problem.

Consider the saddle point illustrated in Figure 1. The dark line down the ‘center’ of Figure 1, marks a 1-dimensional affine subspace within which the gradient never points outside the subspace, at all points within the subspace. Consequently an agent undergoing gradient descent will experience problems in this ‘slice’ of the field if its behaviour satisfies both of the following properties.

- A** The agent starts at, or has reached a position *precisely* in this problematic reduced-dimensionality subspace (being nearby is insufficient).
- B** The agent is able to evaluate and follow the gradient perfectly, e.g. without movement errors or miscalculation of the gradient value to perturb the agent.

If A and B are satisfied, then the agent is trapped in a loop of moving backwards and forwards, os-

cillating around what is in these circumstances a lower-dimensionality local minimum. This lower-dimensionality minimum can be present even in traditionally ‘local-minimum-free’ fields, as it is not associated with a local minimum in the space of the entire surface.

Figure 1 demonstrates that such local minima can be found hidden within a 1-D subspace of the original 2-D map. The minimum is hence within a 2-D subspace of the original potential field’s 3-D space. We call this hidden local minimum problem the *saddle minimum problem*.

It can be seen that if the agent were moving imprecisely across the field, or if the mathematical representation of the potential field was imperfect, then the resulting tiny deviation from the subspace that would be caused would be sufficient to ensure the agent was not trapped in this 2-D saddle minimum. It would instead safely slide down one side of the saddle point and then away from the dangerous area.

The phenomenon described here is not the same as that of *stagnation*, where the field around a saddle point (or group of saddle points) can have such a slight gradient that an agent moving at a rate proportional to the strength of the local gradient proceeds to navigate only very slowly. Connolly points out in (Connolly and Grupen, 1992) that the properties of saddle points in creating flat, slow-progress regions as troublesome, and recommends the use of high precision calculations to deal with the problem of flat regions on harmonic fields. Unfortunately, this recommendation could potentially increase the possibility of saddle minimum problems - see requirement B above. In the case when a saddle point minimum is present and affects the agent, they halt further progress completely by causing oscillation, rather than merely slowing down progress.

It is also important to notice that we are not merely suggesting saddle point positions are a problem; this observation has been made in potential field research already, albeit briefly. Connolly observes in (Connolly et al., 1990) that using a saddle point as a starting point for steepest descent based navigation on a harmonic potential field will result in failure, and recommends using some other local search technique to find an escape route from such positions. On fields possessing saddle minima as described in this paper, a starting point anywhere on the subspace will result in failure in certain descent regimes - so an entire axis of a problem space can be dangerous, not just the saddle point position itself.

We are suggesting the following novel observations:

- Saddle points on navigational potential fields can be prone to the saddle minimum problem.
- When the problem is present, an entire lower-

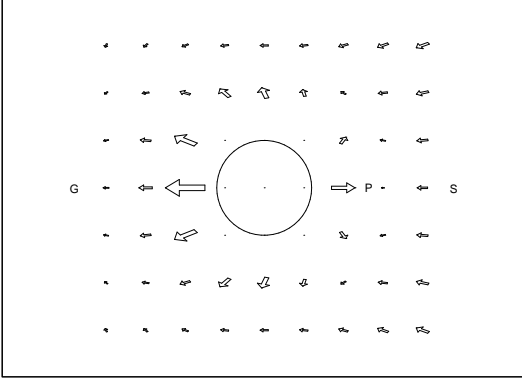


Figure 2: One goal, one obstacle navigation. This potential field contains a single goal  $G$  at  $(-4, 0)$  and a single circular obstacle centered on  $(0, 0)$ . The starting position  $S$  is at  $(4, 0)$ .  $P$  is a saddle point minimum.

dimensional subspace slice of a navigational field is dangerous, not just the saddle point itself, as the subspace contains what is effectively a full-blown local minimum.

- It is plausible that travel may be accidentally or deliberately constrained to this dangerous subspace of the field, e.g. by choice of starting location and descent heuristic.
- Techniques that aim to detect local minima are unlikely to be aware of this type of hidden local minimum, and may be susceptible to such. Techniques that expect no local minima to be present at all may be equally confounded.

### 3. Examples

The following examples are given to demonstrate how easily the saddle minimum problem can occur in a potential field. Any larger environment containing a situation resembling these example problems as a subproblem may well contain the saddle minimum problem as a result.

#### 3.1 Simple 1-Goal Robotic Navigation.

Consider a robot starting at  $S(4, 0)$ , and attempting to navigate to  $G(-4, 0)$ , with an obstacle at  $(0, 0)$ . This is shown in Figure 2. The equation for this field is  $P(x, y) = f(x, y) + g(x, y)$ , where  $f$  and  $g$  are the obstacle and goal functions respectively<sup>1</sup>.

The potential field as a whole is local-minimum-free. It does not contain any local minima, and there is only a single saddle point in front of the obstacle. The fact that this saddle point position would not normally be

<sup>1</sup>The obstacle's function is  $f(x, y) = B/((x-Ox)^2 + (y-Oy)^2)$ , and the goal's function is  $g(x, y) = A*(x-Gx)^2 + (y-Gy)^2$ . Values of  $A = 1, B = 80, Gx = -4, Gy = 0, Ox = 0, Oy = 0$  were used here.

considered as a local minimum can be observed by noting that the arrows at most positions around the saddle point (marked  $P$ ) do not direct the agent back towards  $P$  - rather they direct the agent away from  $P$ .

Looking along the entire  $y = 0$  axis however, we can see that the steepest gradient never has any component along the  $y$  axis. In other words, an agent following the gradient accurately, once placed on this axis, might as well have been placed on a 1-D potential field consisting only of the  $y = 0$  axis. The rest of the potential field has become completely inaccessible because of the combination of starting location, descent heuristic and the shape of the potential field.

#### 3.2 A Historical Problem - Simple 2-Goal Navigation.

Buridan, a 14th Century French philosopher interested in free-will, was unfortunate enough to have a philosophical problem embarrassingly named after him by his critics - 'Buridan's Ass'. The problem consists consists of a completely rational ass (donkey) that is placed at a midpoint between two equally desirable and accessible bales of hay. As both bales of hay are equally attractive, the unfortunate ass proceeds to starve to death, as it cannot rationally pick one bale rather than the other as being best to start moving towards towards.

As well as presenting an excellent metaphor for the theoretical difficulty of solving symmetric problems with deterministic computation, by extending 'Buridan's Ass' we can have an excellent intuitive model of a situation where an agent will fail to navigate after being presented with a potential field with a saddle point and no local minima. Consider an ass that is placed some distance from two bales of hay - and is positioned directly between the two bales of hay as shown in Figure 3. Each bale of hay acts as an attractor for the ass.

The ass will initially move forwards, as this improves its ability to reach either bale of hay. However, it is under no pressure to make a movement preferentially to either side, as the attraction of each bale is balanced by the other bale. Eventually it will reach a point where moving either forwards or backwards will cause the ass to be further away from the bales. Since there is no reason for it to preferentially move towards the left bale of hay, or the right bale of hay, and since it cannot move forwards or backwards except perhaps by a single step, the ass will stop at (or oscillate around) this point and starve to death. This location in the environment is an example of a saddle point minimum - a local minimum, hidden in a lower-dimensionality subspace of the environment. The ass is confined to this subspace by an unfortunate combination of potential field, starting position and descent heuristic.

A real world agent (such as a robot or an ass) that navigates itself towards a saddle minimum will find it-

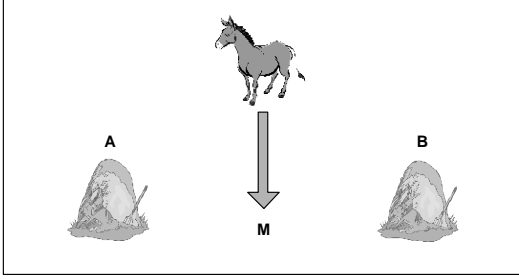


Figure 3: Two alternative goals navigation problem. If bale A and bale B act as attractors for a simulated donkey, it will walk towards both bales of hay simultaneously by travelling towards point M, improving its proximity to each bale by doing so. At point M (a saddle minimum), it can no longer make progress, and since it has no reason to prefer moving towards bale A or bale B, the gradient leaves it trapped, oscillating unhappily in the 1-D hidden local minimum problem in the axis containing the donkey and the midpoint.

self very slightly on one side of the saddle minimum or the other, and thus will be able to continue navigation despite saddle minima. The ‘fuzziness’ of the real world ensures that the agent is never trapped perfectly within the lower-dimensionality subspace.

In philosophical debates (or computer simulations) however, it is quite easy for the agent to carry out navigation entirely within an infinitely thin 1-D slice of the world, and thus the agent will find itself trapped in a local minimum problem in its travel subspace, despite the apparent absence of any local minima in the space of the whole environment.

It can be observed that the potential field the agent can *potentially* travel on is different to the lower-dimensionality potential field that the agent *actually* travels on.

### 3.3 Possible Objections

This section considers some responses objecting to the requirements necessary for the saddle minimum problem to exist. The foci of this paper are experimental simulation of robotic navigation heuristics, and transferral of robotic navigation heuristics to virtual world problem domains. Accordingly, corresponding responses to each objection are given.

#### 3.3.1 “Requirement A) is unlikely. The agent is unlikely to ever intersect perfectly with the subspace.”

In experimental simulation, the most likely cause of the simulated robot landing perfectly on the problem subspace is if the robot is placed there directly by a human favouring an elegant-looking, symmetric problem

design. Examples 1 and 2 are perfect demonstrations of this occurring. Without knowing anything about saddle minima, an obvious test environment for a navigation heuristic is a single goal, and a single circular or point obstacle directly in the way. For 2-goal navigation, placing the robot somewhere between the two goals it has a choice of navigating towards is also an ‘intuitive’ choice.

Similarly, in the case of virtual world agent navigation, it is possible to envisage a scenario whereby computer-controlled monsters are initially placed directly behind an obstacle so as to not disruptively emerge from nowhere in front of a computer game player.

In both simulation and virtual world situations, consider what might happen if the environment is represented by a grid rather than a continuous environment, where the potential and gradient for a whole grid square is chosen according to a sampled point. If the grid sampling points happen to lie on the problem subspace, then the effective ‘problematic subspace’ might be extended to become an entire area on the potential field rather than just a line - and intersection with the subspace during travel becomes a serious possibility.

#### 3.3.2 “Requirement B) is unlikely. No agent can follow the gradient perfectly.”

This depends upon the accuracy of the simulation. In the physical world, it is almost impossible to believe that a robot could travel down this infinitely thin fragment of the environment.

In simulation though, it is possible that this could occur. A floating point real number representation of the environment, and the positions and gradients within it could well introduce sufficient ‘fuzziness’ to ensure that an agent does not stay within the subspace. On the other hand it is equally possible (if the simulation is implemented in a mathematical environment such as Maple or Matlab) that the position of the agent may be tracked with perfect precision by the system’s perfect accuracy (non-floating-point) real number representation.

In a virtual world, it might seem far less likely that a perfectly accurate number system will be used. Most virtual scenarios aim for speed of simulation, and it seems unlikely that an environment such as Maple or Matlab might be used.

It seems possible though that in both scenarios (simulation and virtual world), the number *zero* might be accurately represented by most systems, and if this is the case then it is quite possible in problem situations such as the two examples in this paper, where the problem subspace lies along the  $x$  or  $y$  axis, that the agent will never gain a position or gradient component with a magnitude other than zero. The robot or virtual agent will thus be completely vulnerable to saddle minima.

Lastly, an environment with a grid representation sim-

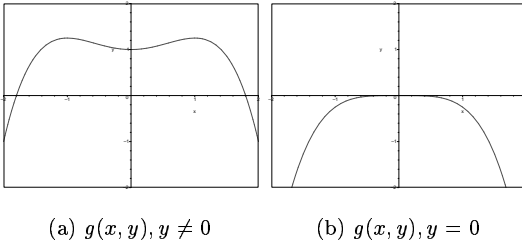


Figure 4: X axis cross sections of the grooved saddle minimum  $g(x, y)$ .

ilar to that detailed in the rebuttal of objection 1 might well make it very easy for a simulated robot or virtual world agent to be kept ‘on track’ even in the event of small floating point errors - i.e. if the gradient is being calculated from the current co-ordinate cell’s midpoint, and the cell’s midpoint lies on the problematic subspace.

### 3.4 Increasing the Threat: Grooved Saddle Minima

Finally, to address any remaining concerns of those still skeptical about the chances of running into this threat, we note a situation under which the danger posed by saddle minima significantly increases. This further variation on the core problem allows the threat area of a saddle minimum to be of the same dimensionality as the problem environment.

Referring back to Figure 1, let the surface function be  $g(x, y)$ , with the saddle point at  $(0, 0)$  and the marked U-shaped ridge running along the  $y$ -axis. Imagine now that this ridge is in fact a groove, or runnel, tapering to zero width at the origin. Thus, an  $x$ -direction cross section of the ridge at some  $y \neq 0$  has the form shown in Figure 4(a) while the cross section at  $y = 0$  has the form shown in Figure 4(b).

A specific example is furnished by the surface  $g(x, y) \equiv -(1/4)x^4 + y^2((1/2)x^2 + 1)$ , a cross section of which has stationary points at  $y, -y$  and  $0$ . Figure 5 shows this function.

Any starting point within either side of the groove will lead to the saddle point, and any overshoot past the saddle point can lead into the corresponding groove on the other side of the saddle point. The threat area of the saddle minimum now has the same dimensionality as the search space itself and continues to have a mathematical identity that is quite distinct from a traditional local minimum.

## 4. Consequences

The authors suspect the consequences of the saddle minimum problem (and grooved saddle minimum) will be

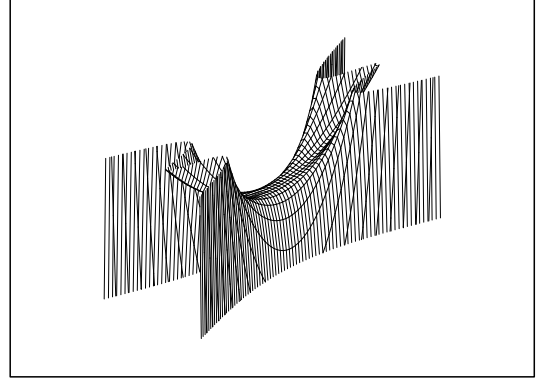


Figure 5: 3-D plot of the grooved saddle minimum  $g(x, y)$ .

generally minor compared with the more common LMP. In a few experiments with simple cases of navigation, the agent or simulated robot will not succeed. These cases will be of the most consequence when e.g. a navigation heuristic is found to have ‘mysteriously’ failed in exactly one case in 100 trials from different starting positions. The problem is less serious in the case of video game or cinematic virtual worlds, where small glitches are less likely to occur than in simulated robotic experiments (due to low precision calculations), and more likely to be tolerated if they occur.

Inverse kinematics problems involving high dimensional configuration spaces are one experimental field of robotics where the saddle minimum problem may be of consequence. Consider the situation of some potential field heuristic being used to navigate a simulated robotic arm between positions in the configuration space of, say, a nine joint-angle effector. The 9-D space in which navigation is occurring is far more difficult to visualise than that of a 2-D environment, and it seems reasonable to suggest that if a saddle minimum is being encountered and causing navigation towards a configuration to fail, it would not be spotted so easily as it would be in a 2-D or 3-D scenario.

It also seems reasonable to consider that such a simulated effector may start off with some joint angles set at ‘zero’. The possibility of initially zeroed variables increases the chance of perfect representation of position and gradient (see Rebuttal 2 in the previous section), and thus increases the chance of saddle minimum requirement B being fulfilled and hence the chance of the problem occurring.

It is also possible that even with imperfect position and gradient representation, a simulated or virtual world agent taking large fixed-size steps over the field will expend most of its efforts upon oscillating around a saddle minimum position and only gradually moving down the side of the saddle, slowing travel considerably. Slowing of progress is not a concern in this paper, however - only the progress-halting behaviour of this hidden analogue

to the traditional LMP.

Finally, the authors note that the saddle minimum problem may affect both traditional and harmonic potential field models (e.g.  $f(x, y) = x^2 - y^2$  satisfies the Laplace equation). The grooved saddle minimum problem could only occur in non-harmonic modellings, as the Laplace equation would prohibit a groove shape from narrowing into a saddle point in this way.

## 5. Possible Solutions

- *Introduce noise.* Suggested in the field of robotics in (Latombe, 1991), this solution is an excellent way of overcoming many problems caused by saddle points. Random or pseudo-random perturbations can be introduced to the position or gradient information to ensure that travel is not restricted to a lower-dimensional subspace, as would be possible with accurate position and gradient information. This solution removes the main threat of the saddle minimum problem - if implemented.
- *Improve LM checking.* Techniques that aim to overcome local minima by detecting them and either avoiding them, or backtracking from them, can add additional tests to check for saddle minima, if there is a chance they might pose a particular problem for some sets of experiments.
- *Environment design.* Picking non-symmetric environments in robotic simulation allows simulated experimental conditions to more closely resemble those in the real world, where perfect symmetry on a macro-scale is impossible to find. The saddle minimum problem, as an artefact entirely due to simulation, is a reminder of the usefulness of Brooks' suggestion to test and develop robotics techniques using real world robots (Brooks, 1991). Using computer generated scenarios and avoiding human-picked scenarios where possible may lead to less problem-causing symmetry occurring in experimental setups.
- *Avoid steepest descent.* Another alternative is to select a better descent heuristic than steepest gradient descent. An underlying problem is that whenever local minima can occur, gradient descent can fail. Some alternative techniques for potential surface travel are less vulnerable to local minima and can be used instead: (XiYong and Jing, 2003) (Bell and Weir, 2004).

## 6. Conclusions.

This paper has demonstrated a type of local minimum problem that can occur on surfaces that are traditionally considered local-minimum-free. These saddle minima may not currently be recognised by techniques that

analyse the local potential field surface to check for local minima. This problem may be of consequence to researchers and students developing potential field navigation heuristics in simulation, or to virtual world designers seeking to re-apply techniques from robotics for virtual world navigation.

Awareness of this problem is useful for those working in potential field based navigation in simulated environments and particularly in highly-dimensional spaces, where the saddle minimum problem may occur in a manner that is extremely difficult to visualise. To overcome the saddle minimum problem, the authors advocate introducing a small amount of random noise to all movement calculations, to remove the symmetry and perfection from the environment that allows the saddle minimum problem to occur.

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